Math 485
Fall 2013
Midterm 1 - Form A
10/10/2013

Name (Print): ____________________

This exam contains 5 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 page of note (1 sided) and a scientific calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. For example, in question involved the multi-period binomial model, I would like to see how you derive the no arbitrage price, say by displaying the tree with all the nodes filled out if the situation is appropriate.
- If the answer involves the probability of a well-known distribution, says the Normal(0,1), **you can leave the answer in the form** $P(Z > x)$ **or** $P(Z < x)$ where x is a number you found from the problem.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	10	
3	10	
4	15	
5	15	
6	15	
7	15	
Total:	100	

1. (a) (6 points) You visit your friend's family, who has 3 children. Upon arriving at the house, you see two girls playing at the front yard. John, your friend, tells you that those are his two daughters. What, then, is the probability that his youngest child is a girl? (Assuming that a boy and a girl are equally likely).

   Ans: This is asking for $P(B|A)$ where $A = \{GGB, GBG, BGG, GGG\}$ and $B = \{BGG, GBG, GGG\}$. So the probability is $\frac{3}{4}$.

   (b) (7 points) The arrival time of Rutgers shuttles to the Hill Center's bus stop from 5:00pm to 5:30pm on a weekday is uniformly distributed. In other words, let X be the waiting time (in minutes) after 5:00 pm until the arrival of a particular shuttle, then X has the p.d.f

$$f_X(x) = \frac{1}{30}, 0 \leq x \leq 30. \tag{1}$$

   Suppose there are 30 shuttles arriving at the Hill Center from 5:00 pm to 5:30 pm, and their distribution are i.i.d. What, then, is the approximate probability that their average arrival time on a particular weekday is before 5:12 pm?

   Ans: Let $X_i, 1 \leq i \leq 30$ be i.i.d. with uniform $[0, 30]$ distribution. Then $\mu := E(X_i) = 15$ and $\sigma^2 := Var(X_i) = 300 - 15^2 = 75$. By the CLT,

$$\frac{\sqrt{n}(\overline{X} - \mu)}{\sigma} \approx N(0, 1),$$

   or

$$\overline{X} \approx Z\frac{\sigma}{\sqrt{n}} + \mu,$$

   where Z has $N(0, 1)$ distribution. So

$$\begin{aligned} P(\overline{X} \leq 12) &\approx P(Z \leq \frac{(12 - \mu)\sqrt{n}}{\sigma}) \\ &\approx P(Z \leq \frac{-3\sqrt{30}}{\sqrt{75}}) = P(Z \leq -3\sqrt{\frac{2}{5}}) \end{aligned}$$

   (c) (7 points) Completing the Financial Math program at Rutgers has a positive effect on students' placement. A student who has succesfully completed the program has a 70% probabilty of landing a job with Goldman Sachs. A student who did not succesfully complete the program, however, only has a 40% probability of landing such a job. Your friend, Tom, just got a position with Goldman Sach. Suppose the probability of a student successfully completing the Financial Math program at Rutgers is 80%. What is the probability that Tom succesfully finished his Financial Math program at Rutgers?

   (Draw a probability tree to illustrate this situation). The answer is

$$\frac{(.8)(.7)}{(.8)(.7) + (.2)(.4)} = .875$$

   Approach using Bayes' forumula: Let A be the event that a person successfully completed the Financial Math program. Then $P(A) = 0.8$. B be the event that a person lands a job with Goldman. Then $P(B|A) = 0.7$ and $P(B|A^C) = .4$. We are asked to find $P(A|B)$.

By Baye's formula:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

Now $P(B) = P(B|A)P(A) + P(B|A^C)P(A^C) = (0.7)(0.8) + (0.4)(0.2)$. And so the answer is $\frac{(.8)(.7)}{(.8)(.7)+(.2)(.4)} = .875$.

2. (10 points) A stock evolves according to the following model:

$$\begin{aligned} S_0 &= 10; \\ S_{i+1} &= S_i X_i. \end{aligned}$$

where X_i are iid random variables with distribution

$$\begin{aligned} X_i &= \frac{4}{3} \text{ with probability } 0.4 \\ X_i &= \frac{1}{3} \text{ with probabiltiy } 0.6. \end{aligned}$$

A financial derivative based on S with expiration time $n = 7$ pays $(S_7 - 18)^2$ upon expiration. What is the price for this financial derivative at time $n = 3$? (Take the interest rate r to be 0.)

Ans: The risk neutral probability is

$$q = \frac{1 - \frac{1}{3}}{\frac{4}{3} - \frac{1}{3}} = \frac{2}{3}.$$

So

$$E^Q(X_i^2) = \left(\frac{4}{3}\right)^2 \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right) = 4.$$

The price for the derivative is

$$\begin{aligned} E^Q((S_7 - 18)^2|S_3) &= E^Q((S_7)^2 - 36S_7 + 18^2|S_3) \\ &= (S_3)^2\big(E^Q(X_1^2)\big)^4 - 36S_3 + 18^2 \\ &= 256(S_3)^2 - 36S_3 + 18^2. \end{aligned}$$

In Questions 4,5 and 6, consider the multiperiod Binomial model where $u = \frac{4}{3}, d = \frac{2}{3}, r = 0, S_0 = 81$.

3. (10 points) Compute the price of V_0 the American put option with strike price $K = 100$ and expiration time $n = 4$.

   Ans: $V_0 = 35.75$

4. (15 points) An Up and In European Call Option with barrier $R = 90$, strike price $K = 100$ and expiration time $n = 3$ pays the option holder the amount $(S_3 - 100)^+$ at time 3, under the additional condition that the stock price has gone above 90 at some time between 0 and 3. (So that if the S always stays below $R = 90$ then the option is worthless). Compute the price at time 0 of this option.

   Ans: $V_0 = 11.5$.

5. (15 points) A Bermudan option gives the option holder the early exercise right before the expiration time, like the American option, except that in a Bermudan option, the holder can only exercise the option at certain times. Compute the price of a Bermudan option for strike price $K = 100$, expiration time $n = 4$ and the only allowed early exercise times are $k = 1$ and $k = 2$.

   Ans: $V_0 = 16.75$

6. In the following, decide whether the process S_k is a martingale, sub-martingale or super-martingale or none of the above with respect to its own filtration $\{\mathcal{F}_k^S\}$ under the probability P.

   (a) (10 points)

   $$S_k = e^{\sum_{i=0}^{k} X_i},$$

   where X_i are i.i.d. with distribution $P(X_i = 2) = \frac{2}{3}$ and $P(X_i = -3) = \frac{1}{3}$.
   Ans: Using the usual technique, we get

   $$E^P(S_{k+1}|\mathcal{F}_k^S) = S_k E^P(e^{X_{k+1}}).$$

   Now

   $$E^P(e^{X_{k+1}}) = \frac{2}{3}e^2 + \frac{1}{3}e^{-3} > 1 \text{ by a calculator.}$$

   So

   $$E^P(S_{k+1}|\mathcal{F}_k^S) > S_k,$$

   S_k is a sub-martingale.

   (b) (5 points)

   $$S_k = e^{\sum_{i=0}^{k} X_i - \frac{k}{2}},$$

   where X_i are i.i.d. Normal (0,1). Recall that Normal(0,1) has the following p.d.f:

   $$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, x \in (-\infty, \infty)$$

   and in general Normal(μ, σ) has the following p.d.f.:

   $$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}, x \in (-\infty, \infty).$$

   Ans: We compute $E(e^Z)$ where Z has $N(0,1)$ distribution

   $$\begin{aligned} E(e^Z) &= \int_{-\infty}^{\infty} e^x e^{\frac{-x^2}{2}} dx \\ &= \int_{-\infty}^{\infty} e^{-\frac{x^2-2x+1}{2}+\frac{1}{2}} = \int_{-\infty}^{\infty} e^{-\frac{(x-1)^2}{2}} e^{\frac{1}{2}} dx. \end{aligned}$$

   Recognizing that $e^{-\frac{(x-1)^2}{2}}$ is the density of a $N(1,1)$ distribution, we get $E(e^Z) = e^{\frac{1}{2}}$.
   Note also that

   $$e^{\sum_{i=0}^{k} X_i - \frac{k}{2}} = e^{\sum_{i=0}^{k-1} X_i - \frac{k-1}{2} + X_{k+1} - \frac{1}{2}} = S_k e^{X_{k+1} - \frac{1}{2}}.$$

Therefore

$$E(S_{k+1}|\mathcal{F}_k^S) = S_k E(e^{X_{k+1}-\frac{1}{2}}) = S_k.$$

Thus S_k is a martingale.

7. (15 points) Consider the one period model: we have a stock whose price at time $k = 0$ is S_0 and

$$\begin{aligned} S_1 &= S_u \text{ with probability } q \\ &= S_d \text{ with probability } 1-q, \end{aligned}$$

where $S_d \le e^{r\tau} \le S_u$. We add an additional assumption that this stock pays dividend at rate q that is reinvested: one share of S at time $k = 0$ is worth $e^{q\tau}S_1$ at time $k = 1$. What is the no arbitrage price of a financial derivative that is worth V_u when $S_1 = S_u$ and worth V_d when $S_1 = S_d$? (Your answer will be in terms of $q, r, \tau, S_0, S_u, S_d, V_u, V_d$). Note also that no arbitrage condition actually requires $e^{q\tau}S_d \le e^{r\tau} \le e^{q\tau}S_u$.

Ans: We construct a replicating portfolio π with δ shares of stocks and b dollars in the money market at time 0. Then

$$\begin{aligned} \pi_0 &= \delta S_0 + b, \\ \pi_1 &= \delta e^{q\tau} S_1 + b e^{r\tau}. \end{aligned}$$

(Note the coupon $eq\tau$ part.) We set $\pi_1 = V_1$ because the portfolio is replicating. So we get the following system of equations for δ and b:

$$\begin{cases} \delta e^{q\tau} S_u + b e^{r\tau} = V_u \\ \delta e^{q\tau} S_d + b e^{r\tau} = V_d. \end{cases}$$

Solving gives

$$\begin{aligned} \delta &= \frac{V_u - V_d}{e^{q\tau}(S_u - S_d)} \\ b &= e^{-r\tau}\left(V_u - \frac{V_u - V_d}{S_u - S_d} S_u\right). \end{aligned}$$

Since the portfolio is replicating, then

$$V_0 = \pi_0 = \frac{V_u - V_d}{e^{q\tau}(S_u - S_d)} S_0 + e^{-r\tau}\left(V_u - \frac{V_u - V_d}{S_u - S_d} S_u\right).$$